

Hedging Strategies of Financial Intermediaries: Pricing Options with a Bid-Ask Spread

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Abstract

This paper uses a model similar to the Boyle-Vorst and Ritchken-Kuo arbitrage-free models for the valuation of options with transactions costs to determine the maximum price to be charged by the financial intermediary writing an option in a non-auction market. Earlier models are extended by recognizing that, in the presence of transactions costs, the price-taking intermediary devising a hedging portfolio faces a tradeoff: to choose a short trading interval with small hedging errors and high transactions costs, or a long trading interval with large hedging errors and low transactions costs. The model presented here also recognizes that when transactions costs induce less frequent portfolio adjustments, investors are faced with a multinomial distribution of asset returns rather than a binomial one. The price upper bound is determined by selecting the trading frequency that will equalize the marginal gain from decreasing hedging errors and the marginal cost of transactions.

Introduction

Financial intermediaries are similar to other financial and non-financial corporations in that they, too, finance their own acquisition of assets by issuing a variety of liabilities on top of their equity claims. Unlike most other businesses, however, financial intermediaries cre-

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ate and hold separate and distinct assets against specific liabilities. A case in point is the creation of a specialized portfolio to support an option written by the intermediary. This paper addresses an issue that concerns financial intermediaries and academics alike: how the bid-ask price spread in a hedging portfolio affects the price of the option charged by the issuing intermediary. The model offered describes that relationship for a price-taking profit-maximizing intermediary.

In the absence of transactions costs and other imperfections, and given certain assumptions on the distribution of returns, the financial intermediary would be able to perfectly hedge itself by creating a synthetic option. In contrast to forward contracts, a perfect hedge against options is achieved through a dynamic strategy that is the basis for risk-neutral option valuation models. However, those models break down in the presence of transaction costs because a perfect hedge requires continuous adjustment of the portfolio replicating the option, pushing those costs to infinity.

Several authors argue that frequent portfolio adjustments at lower total transaction costs allow only an imperfect hedge, leading to an option price defined within upper and lower bounds [1,3,5,6,7,8,10]. Consistent with that scenario, the intermediary writing an option is faced at any point with a tradeoff between bearing greater costs of hedging errors and increasing transaction costs. Given the cost per transaction, more frequent trading would lead to a smaller hedging error but larger periodic transaction costs—two factors having conflicting effects on the spread between a unique equilibrium price free of the effect of transaction costs, and the upper bound limiting the price charged by the intermediary.

In a recent paper, Boyle and Vorst [1] develop an arbitrage-free valuation model for pricing options in the presence of transaction costs. Their model is related to the binomial model of Cox, Ross, and Rubinstein [3] in showing that arbitrage-free option prices are bounded when the underlying asset's return is binomial and the option-replicating portfolio is adjusted discretely. The model presented here tackles the problem of option pric-

ing by an intermediary in a different way. It is recognized that transaction costs induce less frequent portfolio adjustments and thus a longer trading period. This implies that investors are faced in each trading period with a multinomial distribution of returns rather than a binomial one. The results of Ritchken and Kuo [12] and Levy and Levy [10] are employed to price the option under a multinomial distribution and a bid-ask spread. The results of Boyle and Vorst [1] are extended by allowing the trader to choose the length of the trading interval in the presence of transactions costs. Trippi and Harriff [15] point out that the length of the trading interval is path-dependent. While the pricing of the option in the present model relies on the assumption of a fixed trading interval, the trader is free to reassess the trading interval and revise the option's price at any point.

The next section is devoted to two models of option price bounds in the presence of transaction costs. The binomial model described first is used as a basis for constructing subsequently a less restrictive multinomial model. The multinomial model treats the trading frequency as endogenous, and is independent of the investor's risk preference. The third section contains a simulation using the multinomial model to illustrate the intermediary's choice of an option price which leads to equality between the marginal cost of hedging errors and the marginal gain from reducing transaction costs.

Theory

Consider a price-taking financial intermediary writing options on some financial or real asset. The goal of value maximization is translated to the objective of economizing simultaneously on the cost of hedging errors and transaction costs associated with the process of adjusting the portfolio replicating the option.

Let it be assumed that the rate of return earned on the underlying asset's price, S_t , is distributed binomially and there are two states of nature, u and d (see Figure 1). The asset's price is defined as the geometric average of the bid and ask prices, S_{tB} and S_{tA} ,

$$S_t = (S_{tB} \cdot S_{tA})^{1/2} \quad (1)$$

where $S_{tB} = S_t/\alpha$ and $S_{tA} = S_t\alpha$, and $(\alpha^2 - 1)$ is the percentage difference between the bid and ask prices. For simplicity, it is further assumed that, for an institutional investor operating in the inter-bank market, transaction costs consist only of the bid-ask spread of the underlying asset. It is also assumed that the intermediary is a price taker, that proportional unit transaction costs, α , are constant, and that there are n trading periods prior to the expiration of the option. For simplicity it is assumed that the asset pays no dividends.

Binomial Model

Based on Boyle and Vorst [1] and Merton [11], the upper bound of the option price charged by the financial intermediary is found by constructing an arbitrage transaction for a call option. Let the intermediary buy δ units of the underlying asset at the ask price and sell the option, so that the initial cash flow is $-\delta S\alpha + C$, where C denotes the price of the call option. The value of the portfolio in states of nature u and d , respectively, is

$$\text{State } u: \delta S u / \alpha - C_u$$

$$\text{State } d: \delta S d / \alpha - C_d$$

where u and d are the returns on the underlying asset, and C_u and C_d are the values of the call option in period 1, in states u and d respectively. To find the optimal hedge ratio δ , the return in the two states of nature is set equal

$$\delta S u / \alpha - C_u = \delta S d / \alpha - C_d$$

yielding the solution

$$\delta = [\alpha(C_u - C_d)] / [S(u - d)] \quad (2)$$

By buying delta units of the underlying asset and selling (writing) the call option, the intermediary transforms the portfolio into a riskless one, paying out the same amount of money in each of the states of nature. The intermediary will finance the portfolio by borrowing funds at the riskless interest rate, $r - 1$, where r denotes one plus the interest rate.¹ Abnormal arbitrage profits will be present as long as the riskless return on the portfolio is greater than the cost of financing it, a condition occurring if

$$\delta S u / \alpha - C_u - (\delta S \alpha - C) r \geq 0 \quad (3)$$

In equilibrium, the intermediary with the lowest unit transactions costs, α , will set an upper bound to the price of the call option. To prevent arbitrage, inequality (3) must be reversed

$$\delta S u / \alpha - C_u - (\delta S \alpha - C) r \leq 0 \quad (3')$$

A price upper bound for the call option is derived by substituting (2) in (3') and solving for C

$$C \leq [q C_u + (1 - q) C_d] / r \quad (4)$$

where $q = (\alpha^2 r - d) / (u - d)$ and $0 \leq q \leq 1$.² In order to get a two-period price upper bound, the two possible states of nature in period 1 are denoted by $S_1 = S \cdot u$ and $S_1 = S \cdot d$, where S is the initial price of the underlying asset in period 0. For $S_1 = S \cdot u$ (and $C_1 = C_u$), the one-period analysis is repeated for period 2 given the initial price Su , to yield the upper bound

$$C_u \leq [q C_{uu} + (1 - q) C_{ud}] / r \quad (5)$$

Likewise, the upper bound for C_d is found given the initial asset price Sd

$$C_d \leq [q C_{du} + (1 - q) C_{dd}] / r \quad (5')$$

where C_{uu} , C_{ud} , C_{du} , and C_{dd} are the option prices in period 2 in each of the four possible states of nature.

To find the two-period price upper bound, we substitute (5) and (5') into (4)

$$C \leq [q^2 C_{uu} + 2q(1-q)C_{ud} + (1-q)^2 C_{dd}] / r^2 \quad (6)$$

Based on Cox and Rubinstein [2], the price upper bound at time 0 under the n -period scenario is obtained by continuing the recursive process through the expiration date

$$C \leq (1/r^n) \sum_{j=0}^n \left[\binom{n}{j} q^j (1-q)^{n-j} C(u,j) \right] \quad (7)$$

where $C(u,j)$ is the value of the option at expiration by which u occurs j times and d occurs $n-j$ times, namely

$$C(u,j) = \max[0, Su^j d^{n-j} / \alpha - X]$$

A similar arbitrage transaction is conducted to find the upper bound of a put option price charged by the financial intermediary in the presence of a bid-ask spread: the intermediary sells (shorts) an amount δ' of the underlying asset at the bid rate, sells short (writes) one unit of a put option, P , and lends the proceeds. Construction of this transaction yields the hedge ratio $\delta' = [P_u - P_d] / [S\alpha(d-u)]$. By selling short the amount δ' , the intermediary ensures that the portfolio will yield the same return in both states u and d . The price upper bound of a put option at time 0 is

$$P \leq (1/r^n) \sum_{j=0}^n \left[\binom{n}{j} q'^j (1-q')^{n-j} P(u,j) \right] \quad (8)$$

where $q' = [r - d\alpha^2] / [\alpha^2(u-d)]$ and $0 \leq q' \leq 1$.³ The value of the option at expiration by which u occurs j times and d occurs $n-j$ times is

$$P(u,j) = \max[0, X - Su^j d^{n-j} \cdot \alpha]$$

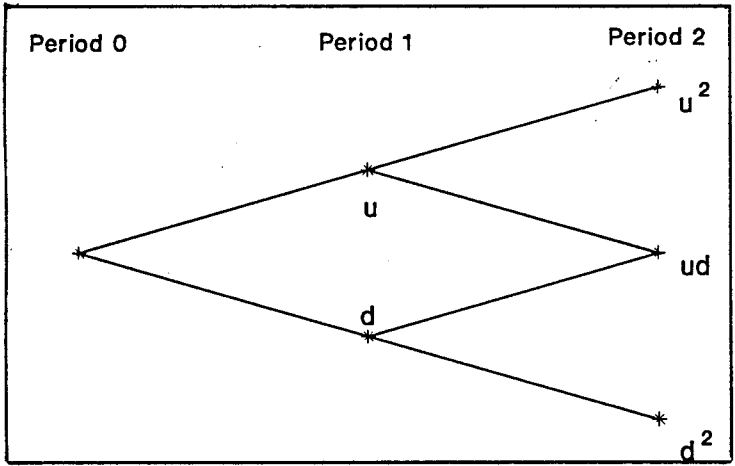
The interpretation of these results is straightforward. The combination of binomial asset returns and transaction costs leads the price-taking intermediary to charge a price which is higher than the price predicted by Black and Scholes in the absence of transaction costs. The increased price must cover the intermediary's costs of transactions and hedging errors. Given that an increase in the frequency of transaction (i.e., a decrease in the trading interval) decreases hedging errors, the optimal trading interval is found where the marginal cost of transactions equals the marginal gain from decreasing hedging errors. This problem is tackled next.

Multinomial Model

An obvious limitation inherent in the binomial model employed by Merton [11] and Boyle and Vorst [1] is the assumption that the investor must trade in every binomial period. Under the less restrictive assumption that the intermediary can choose the trading interval, each trading interval may contain one or more periods of binomial returns, so that the distribution of the underlying asset in each trading period becomes multinomial (see Figure 1a).

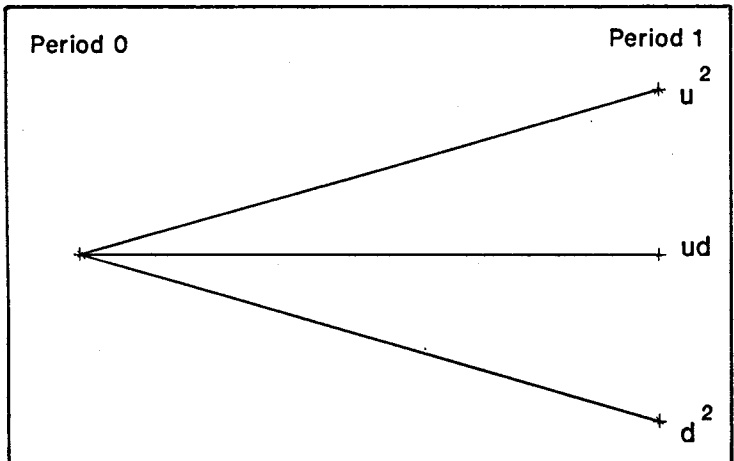
Like the binomial model, the multinomial one is based on the assumption that the true periodic distribution of the underlying asset is binomial with n binomial periods prior to the option's maturity. Since the intermediary may avoid trading in every binomial period to economize on transaction costs, the distribution in each trading period becomes multinomial. To illustrate this point, assume that there are two binomial periods to expiration, and the intermediary decides to trade in both. As shown in Figure 1a, the intermediary is faced with two possible states of nature, u and d , in the first period and three possible states, u^2 , ud and d^2 , in the second period. However, if the intermediary decides to trade once every two binomial periods, then it is faced with three possible states of nature after one trinomial period. If the underlying distribution is binomial, the three states of nature would also be u^2 , ud and d^2 (Figure 1b).

Figure 1a
Illustration of Binomial Distribution



Two Trading Intervals

Figure 1b
Illustration of Multinomial Distribution



Single Trading Interval

Suppose there are n upward or downward binomial moves until expiration, and the intermediary decides to trade m times where $m < n$. Thus

$$h = n/m \quad (9)$$

is the length of the actual trading period. By extrapolating Figure 2, it can be shown that the distribution of returns becomes multinomial with 2^h states (since some of the states coincide, the number of unique states is only $h + 1$).

Ritchken and Kuo [11] derive a price upper bound in the multinomial case without transactions costs.⁴ They assume that there are h possible states of nature in each trading period, with $u_1 > u_2 > \dots > u_h$, and that the arbitrage-free option price upper bound is given by the binomial pricing formula when u_1 is substituted for u and u_h for d . To assimilate transaction costs in these results, Define $u_h = u^h$ and $d_h = d^h$ where, respectively, u_h and d_h are the maximum and minimum returns on the underlying asset over trading period h . Notation r_h stands for one plus the interest rate over the same trading period (i.e., $r_h = r^h$). Thus, the call option price upper bound is

$$C \leq (1/r_h^m) \sum_{j=0}^m \left[\binom{m}{j} q_h^j (1 - q_h)^{m-j} C(u_h, j) \right] \quad (10)$$

where $q_h = (\alpha^2 r_h - d_h) / (u_h - d_h)$, and $C(u_h, j) = \max[0, S u_h^j \times d_h^{m-j} / \alpha - X]$ is the value of the call option at expiration by which u_h occurs j times and d_h occurs $m - j$ times.

Similarly, the put option price upper bound is

$$P \leq (1/r_h^m) \sum_{j=0}^m \left[\binom{m}{j} q_h'^j (1 - q_h')^{m-j} P(u_h, j) \right] \quad (11)$$

where $q_h' = (r_h - d_h \alpha^2) / [\alpha^2 (u_h - d_h)]$, and $P(u_h, j) = \max[0, X - S \cdot u_h^j d_h^{m-j} \cdot \alpha]$ is the value of the put option at expiration by which u_h occurs j times and d_h occurs $m - j$ times.

Although outside the scope of this paper, these results can also be used to price options purchased by the intermediary, leading to a lower bound of the option price.

Simulation

Next, the multinomial model is employed to simulate the upper bound for the price charged by the financial intermediary on an option written on the asset, allowing the arbitrageur to choose the optimal combination of trading interval and transaction costs. For simplicity, the simulation adheres to the assumption that all transaction costs are embedded in the bid-ask spread of the underlying asset. As shown above, for any combination of unit transaction costs, α , and trading interval, h , it is possible to compute a price upper bound defined as the maximum option price preventing arbitrage. To compare the proposed model with that of Black and Scholes, the approach of Cox and Rubinstein [2] is followed by setting $u = \exp(\sigma\sqrt{dt})$, $d = 1/u$, and assuming $\sigma = 0.10$. This ensures that the price upper bound based on the present model converges to the Black-Scholes price when transaction costs vanish ($\alpha = 1$) and the replicating portfolio is adjusted every binomial period ($h = 1$). Note that the binomial model is viewed in the literature as an approximation for the Black-Scholes model that does not depend on the number of binomial trading periods for a relatively large n . With the introduction of transaction costs, the assumption regarding n becomes critical. If the arbitrageur trades in every period in the presence of such costs, the spread between the price upper bound and the Black-Scholes price increases with any increase of n .

Figures 2a and 2b illustrate side by side the Black-Scholes price and the price upper bound derived by the multinomial model as a function of the trading interval, h , for alternative feasible values of the transaction costs parameter, α , using the following additional parameters: $S = 100$; $X = 100$; $T = 0.25$; $n = 180$ or $n = 90$ (held constant); $h = 1, 2, \dots, 6$; and $r = 0.1$.

The following main features of the proposed multinomial model can be garnered from Figure 2a ($n = 180$)

Figure 2a
 A Call Price Upper Bound
 ($S=X=100$; $T=0.25$; $r=0.1$; $\text{Sig}=0.1$; $n=180$)

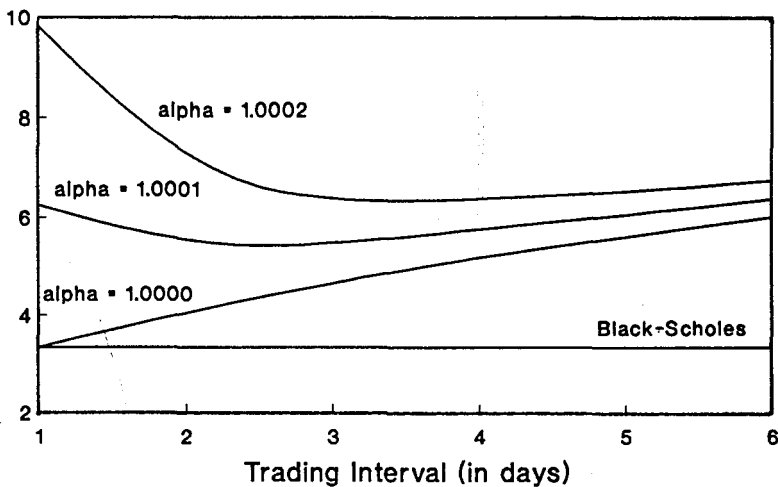
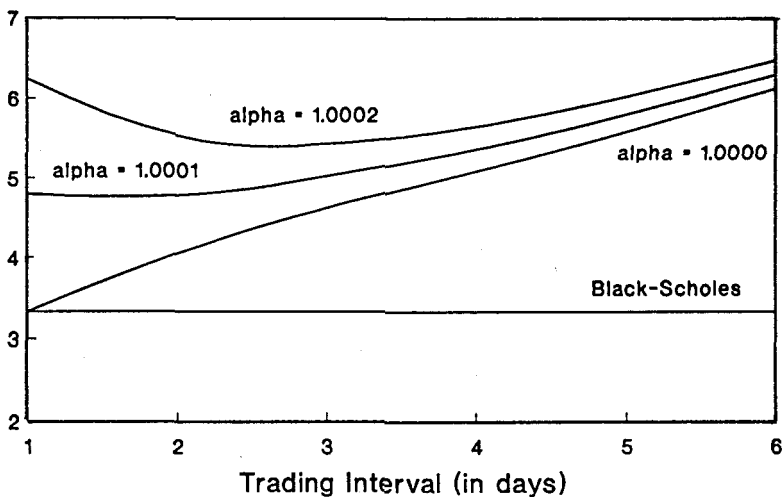


Figure 2b
 A Call Price Upper Bound
 ($S=X=100$; $T=0.25$; $r=0.1$; $\text{Sig}=0.1$; $n=90$)



and Figure 2b ($n = 90$). First, the optimal call option price is equal to the Black-Scholes price only in the special case of no transaction costs, $\alpha = 1$, and trading in every binomial period, $h = 1$. In this case, the trader costlessly creates a perfect hedge by revising the option-replicating portfolio every binomial period. Second, the higher the unit transaction costs, α , the higher is the price upper bound for any given trading interval, h . While in the absence of transaction costs it is optimal to trade every binomial period, less frequent trading is optimal when $\alpha > 1$; trading every binomial period incurs the greatest amount of transaction costs. Third, the longer the trading interval, h , the higher is the price upper bound for any given unit transaction costs, α . Fourth, both higher transaction costs and longer trading periods prevent the possibility of constructing a perfect hedge. Fifth, for a trading interval of almost three days, the call price upper bound is lower in Figure 2b than in Figure 2a because of a smaller number of adjustments until expiration time (three months).

Thus, while an increase in the trading interval will raise the price upper bound due to a less perfect hedge, it will also lower the total transaction costs and may make it advantageous not to trade in every binomial period. That is, the trading interval which minimizes the price upper bound may be greater than unity. As seen in Figures 2a and 2b, when $\alpha = 1.0002$ (0.02 percent) the minimum price upper bound is attained approximately at $h = 3$ where trading is conducted once every three binomial periods. In this case, the positive effect of cutting on transaction costs outweighs the advantage of a perfect hedge up to a trading interval of $h = 3$. This relationship is reversed for $h > 3$.

Summary

This paper uses an arbitrage model for the valuation of options with transaction costs to determine the maximum price to be charged by the financial intermediary writing an option in a non-auction market. Earlier models are extended by recognizing that in the presence of transaction costs, the price-taking intermediary devis-

ing a hedging portfolio faces a tradeoff: to choose a short trading interval with small hedging errors and high transaction costs, or a long trading interval with large hedging errors and low transaction costs. The proposed model also recognizes that when transaction costs induce less frequent portfolio adjustments, investors are faced with a multinomial distribution of asset returns rather than a binomial one. The price upper bound is determined by selecting the trading frequency that will equalize the marginal gain from decreasing hedging errors and the marginal cost of transactions. Although the proposed model relies on the assumption of a fixed trading interval, the investor is free to reassess and readjust the trading interval and option's pricing at any time. This model can be used to price a variety of contingent claims handled by financial intermediaries.

Notes

1. This model can accommodate other contingent claims as well. For example, in the case of foreign currency options, S would be replaced by S/r^* where r^* is one plus the foreign interest rate.

2. The terms d/α^2 and u/α^2 represent the returns in both states of nature from buying the underlying asset at the ask price and selling it at the bid price. The absence of arbitrage indicates $r \geq d/\alpha^2$, and thus $q \geq 0$ since otherwise one could buy the portfolio, borrow at the riskless rate, and have arbitrage profits in all states of nature. Given this result, $q \geq 0$. While theoretically it is possible that $r \geq u/\alpha^2$ and $q \geq 1$, it will occur only if α is large. The authors of this paper preclude this possibility by assuming that α is sufficiently small.

3. The terms $\alpha^2 d$ and $\alpha^2 u$ represent the returns in both states of nature of selling short the underlying asset at the bid price and buying it back at the ask price. Arbitrage is prevented by $r \leq \alpha^2 u$ and thus $q' \leq 1$. Although theoretically it is possible that $r < \alpha^2 d$, the authors of this paper do not expect this to occur if α is relatively small, given the no-arbitrage argument $r \geq d/\alpha^2$.

4. Levy and Levy [10] obtain the same upper bound for continuous distributions as well.

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